Suppose $X_i : f \longrightarrow x_i f$ then $D_i = \partial_i \circ X_i$

Two formulas for Schur Polynomials

$$S_\lambda(x_1, \dots, x_n) = \partial_{w_0}(x^{\lambda+\delta}) \stackrel{?}{=} D_{w_0}(x^\delta)$$

<u>Thm</u> $\partial_{w_0} \circ X_1^{n-1} X_2^{n-2} \cdots X_n^0 = D_{w_0}$

<u>Lemma</u>
- $\partial_i X_j = X_j \partial_i$ if $j \notin \{i, i+1\}$
- $\partial_i (X_i X_{i+1}) = (X_i X_{i+1}) \partial_i$

Checking Lemma and Lemma $\Rightarrow$ Thm is now easy.

<u>proof</u> (Thm)

Consider the following reduced decomposition

$$w_0 = (s_1 \cdots s_{n-1})(s_1 \cdots s_{n-2}) \cdots (s_1 s_2)(s_1)$$

$$D_{w_0} = (\partial_1 X_1 \partial_2 X_2 \cdots \partial_{n-1} X_{n-1})$$
$$(\partial_1 X_1 \cdots \partial_{n-2} X_{n-2})$$
$$\vdots$$
$$(\partial_1 X_1 \partial_2 X_2)$$
$$(\partial_1 X_1)$$

by Lemma, we can move X-es to the right in each brackets to get

$$D_{w_0} = (\partial_1 \partial_2 \cdots \partial_{n-1} X_1 \cdots X_{n-1})$$
$$(\partial_1 \partial_2 \cdots \partial_{n-2} X_1 \cdots X_{n-2})$$
$$\vdots$$
$$(\partial_1 \partial_2 X_1 X_2)$$
$$(\partial_1 X_1)$$

Now we can move $X_1 X_2 \cdots X_k$ together to the right for all $k = 1, 2, \ldots, n-1$. $\square$

- Can we do similar procedure for all perm-s?
- Turns out that no.

↖ You can do if and only if permutation is 312 avoiding.

This is related to

→ Schubert Polynomials $\mathfrak{S}_u = \partial_{u^{-1}w_0}(x^\delta)$

→ Demazure Char. (Key polynomials)

$$ch_{\lambda,w} = D_w(x^\lambda)$$

$\lambda = (\lambda_1 \geq \dots \geq \lambda_n)$ and $w \in S_n$

<u>Thm 1</u> If $w \in S_n$ is 312-avoiding permutation then $ch_{\lambda,w} = D_w(x^\lambda)$ is a Schubert poly $\mathfrak{S}_u$ for some $u \in S_m$.

<u>Thm 2</u> If $u \in S_m$ is 2143_avoiding perm. then $\mathfrak{S}_u$ is a certain Demasure character $ch_{\lambda, w} = D_w(x^{\lambda})$ for some $w \in S_n$.

2143_avoiding perms. are called <u>vexillary</u>

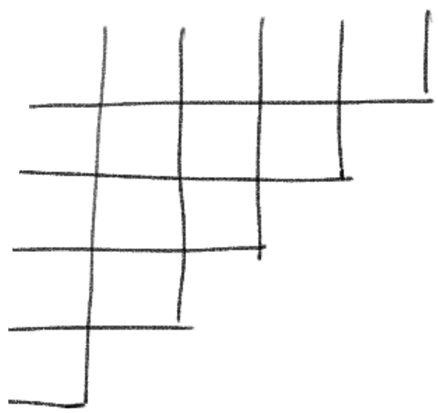
In order to understand these, we need to develop some combinatorics.

$$S_\lambda(x_1, \dots, x_n) = \sum_{\substack{T \text{ is semi-standard} \\ \text{Young Tableau of shape } \lambda}} x^{\text{weight}(T)}$$

Combinatorial Formula for $\mathfrak{S}_w$

RC-graphs aka pipe dreams

[Fomin-Stanley, Billey-Jockush-Stanley]

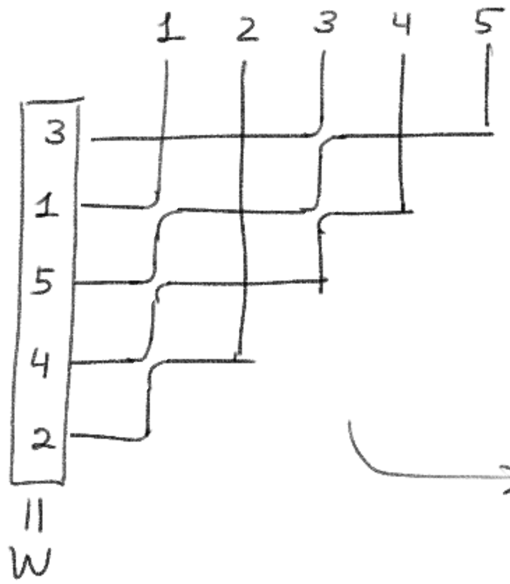


we replace some of the crossings

then we get a wiring diagram

$\longrightarrow$ any two wires intersect at most once

we can read a permutation



weight $(P) = (\beta_1, \beta_2, \dots)$

β_i records # crossings in i^{th} row.

weight $(3,1,1,0,0)$

<u>Thm</u> [FS, BJS]

$$\mathfrak{S}_w = \sum_{P = \text{pipedream for } w} x^{\text{weight}(P)}$$